

Combination Properties of Weakly Contracting Systems

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Abstract

Evaluating the dimension of attractors for autonomous nonlinear dynamical systems has a distinguished history. In this paper we recast these classical results in the context of contraction theory. In particular, we explore the role of Riemannian metrics, quantify convergence rates and the effects of averaging, and derive general system combination properties. Many of these results extend straightforwardly to non-autonomous systems, and suggest potential applications in computational biology and systems neuroscience.

1 Basic results

Consider an autonomous system with state $x \in \mathbb{R}^n$ and dynamics

$$\dot{x} = f(x) \tag{1}$$

We are interested in the limiting behaviour of such a system as $t \rightarrow +\infty$, and how this behaviour is preserved under interconnection. We assume that the system evolves on a compact and simply-connected strictly forward-invariant set \mathcal{X} . That is, any solution starting on the boundary of \mathcal{X} at $t = 0$ remains in the interior of \mathcal{X} for $t \geq 0$. The matrix $J(x) = \frac{\partial f}{\partial x}$ denotes the system's Jacobian, and $J_s(x) = \frac{1}{2}(J(x) + J(x)')$ its symmetric part.

For a symmetric $n \times n$ matrix H , we define $\lambda_1(H) \geq \lambda_2(H) \geq \dots \geq \lambda_n(H)$ as the eigenvalues of H ranked in non-increasing order. We also define a function giving the sum of the k largest eigenvalues:

$$S_k(H) = \sum_{j=1}^k \lambda_j(H)$$

We are interested in systems satisfying the following property:

$$\forall x \in \mathbb{R}^n \quad S_k(J_s(x)) < 0$$

Such a property allows one to bound the Hausdorff dimension of an attractor of (1), and to our knowledge was first investigated by Douady and Oesterlé [1], and subsequently studied by Smith [2] and Leonov and colleagues (see, e.g., [3], [4]). Extensions include non-autonomous systems and the addition of a storage function to reduce conservatism.

In particular, if $k = 2$, i.e. $\lambda_1(J_s) + \lambda_2(J_s) < 0$, then all bounded trajectories converge to an equilibrium (not necessarily unique), i.e. the dimension of any attractor is zero. Following Leonov, we refer to this property of eigenvalues as *weak contraction* and to a system having this property as a *weakly contracting* system.

By contrast, a contracting system (in the identity metric) has the property that $S_1(J_s(x)) < 0$, i.e. the largest eigenvalue $\lambda_1(J_s(x))$ is negative [5].

Convergence to an equilibrium is unchanged by coordinate transformation, so we will also call a system weakly contracting if there exists a constant non-singular matrix Θ such that $S_2(\Theta J(x)\Theta^{-1} + (\Theta J(x)\Theta^{-1})') < 0$ for all x . When we need to be specific, we say that the system is *weakly contracting under T* [4]. More generally, as in [5], one can consider transformations $\Theta(x, t)$, where $\Theta(x, t)^T \Theta(x, t)$ is a uniformly positive definite Riemannian metric, and the system is weakly contracting if $S_2(F_s) < 0$ for all x , where F is the *generalized Jacobian* [5] associated with $\Theta(x, t)$,

$$F = \Theta J \Theta^{-1} + \dot{\Theta} \Theta^{-1}$$

In the yet more general case where the metric transformation $\Theta(x, t)$ is complex, the condition becomes that $S_2(F_H) < 0$ for all x , where F_H is the Hermitian part of the generalized Jacobian F and the Riemannian metric is $\Theta(x, t)^* \Theta(x, t)$.

Note that if it is known that the system has a unique equilibrium and trajectories are bounded, then the condition that $\lambda_1 + \lambda_2 < 0$ in some metric (i.e., $S_2(F_H) < 0$ for some $\Theta(x, t)$) guarantees global convergence to that equilibrium, using a weaker condition than the full contraction condition $\lambda_1 < 0$ (i.e., $S_1(F_H) < 0$) in that metric.

Remark 1. For computational methods, it is important to note that $S_k(H)$ has a representation, due to Ky Fan, as

$$S_k(H) = \max_{V \in \mathcal{V}_k} \text{Tr}(V H V')$$

where \mathcal{V}_k is the set of all $k \times n$ matrices with orthonormal rows, i.e. satisfying $V V' = I_k$. From this construction, it is clear that S_k is convex, since it is the maximum of an infinite family of functions linear in M . In fact, $S_k(H)$ can be represented as a linear matrix inequality [6, p. 238].

Intuitively, in the above equation $V H V'$ is the projection of the symmetric matrix H on an orthonormal basis spanning a k -dimensional subspace of \mathbb{R}^n ,

and the max operation picks the basis aligned with the eigenvectors corresponding to the largest eigenvalues of H .

1.1 Example

A simple damped pendulum

$$\ddot{x} + b\dot{x} + \sin x = 0$$

naturally has a cylindrical phase space with two equilibria, however if the angle x is considered as a real number then there are equilibria at $k\pi$, $k \in \mathbb{Z}$. Defining a state $[x, \dot{x}]'$ yields the Jacobian

$$J = \begin{bmatrix} 0 & 1 \\ -\cos x & -b \end{bmatrix}$$

Since the system is two-dimensional, the condition on the eigenvalues of J_s is directly given by the trace of J_s , which is simply the trace of J ,

$$\lambda_1 + \lambda_2 = -b < 0$$

Hence every system trajectory converges to an equilibrium. Note that any system trajectory starting at one of the unstable equilibria remains there, while any other trajectory converges to a stable equilibrium.

Note that one can easily show first that all system trajectories are bounded, as the theorem requires.

For planar systems, weak contraction is related to the Bendixson criterion for non-existence of limit cycles, but of course weak contraction extends to higher-order systems.

2 Combination Properties

In the spirit of [5], we remark that it is straightforward to show that certain combinations of contracting and weakly contracting systems preserve the weak contraction property, and therefore they have the property that all solutions converge to an equilibrium. We assume that the combination does not result in the system leaving the set on which the weak contraction condition holds.

2.1 Parallel Interconnection

Theorem 1. *Assume that two systems f_a and f_b are weakly contracting under the same metric transformation Θ , then so is the system*

$$\dot{x} = \alpha f_a(x) + \beta f_b(x)$$

where α, β are non-negative constants, and $\alpha + \beta > 0$.

Proof. The symmetric part of the Jacobian is

$$J_s^{ab} = \alpha J_s^a + \beta J_s^b$$

so that $J_s^{ab} < 0 \Leftrightarrow J_s^{ab}/(\alpha + \beta) < 0$. But the latter is a convex combination of J_s^a and J_s^b both of which are negative by assumption, and S_2 is a convex function, $J_s^{ab} < 0$. \square

By recursion, this can be extended to non-negative combinations of any number of weakly contracting systems.

2.2 Skew-Symmetric Feedback Interconnection

Consider a skew-symmetric interconnection of a weakly contracting system f_a and a contracting system f_b , with the Jacobian

$$J^{ab}(x) = \begin{bmatrix} J^a(x) & G(x) \\ -G(x)' & J^b(x) \end{bmatrix}$$

for arbitrary $G(x)$. We define $\lambda_i^a = \lambda_i(J_s^a)$ and likewise for λ_i^b .

Theorem 2. *If $\lambda_1^a + \lambda_1^b < 0$ then the feedback interconnection is weakly contracting.*

Proof. The symmetric part of the interconnection's Jacobian is

$$J_s^{ab}(x) = \begin{bmatrix} J_s^a(x) & 0 \\ 0 & J_s^b(x) \end{bmatrix}$$

and so each eigenvalue $\lambda_i^a, \lambda_j^b, i = 1, 2, \dots, j = 1, 2, \dots$ of $J_s^a(x), J_s^b(x)$ is an eigenvalue of J_s^{ab}

Either $\lambda_1^b \leq \lambda_2^a$ or $\lambda_1^b > \lambda_2^a$. In the first case the combined system has the same $\lambda_1 + \lambda_2$ as system a . In the second case, λ_1^b becomes the new λ_2^{ab} and so $\lambda_1^a + \lambda_1^b < 0$ implies weak contraction. \square

Note from the proof that a less restrictive (although possibly less convenient) test would be that the sum of the two largest of $\lambda_1^a, \lambda_2^a, \lambda_1^b, \lambda_2^b$ should be strictly negative.

Following [7], the result can easily be extended to more complex feedback interconnections by using metric transformations. In particular, it extends immediately to any constant loop gain $k > 0$,

$$J^{ab}(x) = \begin{bmatrix} J^a(x) & k G(x) \\ -G(x)' & J^b(x) \end{bmatrix}$$

by using the metric transformation

$$\Theta = \begin{bmatrix} I_n & 0 \\ 0 & \sqrt{k} I_m \end{bmatrix}$$

where n and m are the state dimensions of systems a and b .

The results extend by recursion to similar feedback combinations of any number of systems.

2.3 Hierarchical Interconnections

Consider a hierarchical connection of systems with the Jacobian

$$J^{ab}(x) = \begin{bmatrix} J^a(x) & G(x) \\ 0 & J^b(x) \end{bmatrix}$$

for arbitrary $G(x)$, with either system a or b weakly contracting, and the other contracting.

Theorem 3. *If $\lambda_1^a + \lambda_1^b < 0$ then the hierarchical interconnection is weakly contracting.*

Proof. Let n and m be the state dimensions of systems a and b . Consider the family of transformations

$$\Theta = \begin{bmatrix} I_n & 0 \\ 0 & \epsilon I_m \end{bmatrix}$$

where $\epsilon > 0$ is a parameter, and I_r is the r -dimensional identity matrix. The symmetric part of the interconnection's Jacobian is

$$J_s^{ab}(x) = \begin{bmatrix} J_s^a(x) & \frac{\epsilon}{2} G(x) \\ \frac{\epsilon}{2} G(x)' & J_s^b(x) \end{bmatrix}$$

and clearly for each x there is a sufficiently small ϵ such that the eigenvalues of J_s^{ab} can be brought arbitrarily close to the eigenvalues of the matrix

$$\begin{bmatrix} J_s^a(x) & 0 \\ 0 & J_s^b(x) \end{bmatrix}$$

and since \mathcal{X} is compact, one can find an ϵ that works uniformly for $x \in \mathcal{X}$. The remainder of the proof follows exactly that of Theorem 2. \square

The remarks following the skew-symmetric interconnection also apply here.

By recursion, any combination of the above combinations yields a weakly contracting system, and therefore every bounded trajectory of the overall system tends to an equilibrium.

3 Extensions

In this section, we consider more generally a *non-autonomous* system

$$\dot{x} = f(x, t) \quad (2)$$

3.1 Exponential convergence of i -dimensional volumes

Proceeding exactly as in [3, 4], one can show

$$\frac{d}{dt} \|\delta z_1 \wedge \delta z_2\| \leq (\lambda_1 + \lambda_2) \|\delta z_1 \wedge \delta z_2\|$$

where each δz now represents a differential displacement weighted by the metric transformation $\Theta(x, t)$, that is $\delta z = \Theta(x, t)\delta x$ [5], the λ_j are the ranked eigenvalues of the symmetric (or Hermitian) part of the generalized Jacobian associated with $\Theta(x, t)$, and \wedge denotes the vector product. The above implies that if $\lambda_1 + \lambda_2$ is upper bounded by some negative constant $-\alpha$, then the *area* of differential surfaces shrinks exponentially with rate α . If this condition on the largest two eigenvalues is verified in some metric, we will also call the non-autonomous system *weakly contracting*.

As in [3, 4], the result extends to sub-volumes of higher dimensions, all the way to the familiar Gauss theorem relating rate of change of volume to system divergence,

$$\forall i = 1, \dots, n \quad \frac{d}{dt} \|\delta z_1 \wedge \dots \wedge \delta z_i\| \leq (\lambda_1 + \dots + \lambda_i) \|\delta z_1 \wedge \dots \wedge \delta z_i\|$$

where \wedge denotes more generally the outer product and $\|\delta z_1 \wedge \dots \wedge \delta z_i\|$ measures the volume of the polytope constructed on the δz_j [8].

Note that in the case of weakly contracting *autonomous* systems, the result in [3] (for identity or constant metric) is more precise – not only all surfaces shrink, but actually any bounded trajectory asymptotically tends to some equilibrium (and not just to some 1-dimensional manifold).

3.2 Storage function

As noticed earlier, [3, 4] also suggest the introduction of a storage function to reduce conservatism and obtain more general bounds. This is also the approach of [9] in the more specific application of convergence to trajectories. In the context of this paper, the same modification can be derived as follows.

First, consider a metric transformation of the simple form $\Theta_e(t) = e^{\gamma(t)} I$, where $\gamma(t)$ is a differentiable scalar function. This leads to the generalized

Jacobian

$$\Theta_e J \Theta_e^{-1} + \dot{\Theta}_e \Theta_e^{-1} = J(x) + \dot{\gamma}(t) I$$

More generally, given an original $\Theta(x, t)$, an augmented metric transformation of the form

$$\Theta_e(x, t) = e^{\gamma(x, t)} \Theta(x, t)$$

yields an extra term $\dot{\gamma} I$ in the generalized Jacobian F_e ,

$$F_e = \Theta_e J \Theta_e^{-1} + \dot{\Theta}_e \Theta_e^{-1} = \Theta J \Theta^{-1} + \dot{\Theta} \Theta^{-1} + \dot{\gamma} I = F + \dot{\gamma} I$$

Assume now that the function $\gamma(x, t)$ we chose is *bounded*, similarly to [3, 4, 9]. By construction, letting

$$\delta z = \Theta_e \delta x$$

we have

$$\frac{d}{dt} \delta z \leq \lambda_1(F_e) \delta z$$

so that

$$\forall t \geq 0 \quad \|\delta z(t)\| \leq \|\delta z(t=0)\| e^{\int_0^t \lambda_1(F_e)(t) dt}$$

Since $\gamma(x, t)$ is bounded, in the exponential above it will be dominated by the original contraction rate,

$$\int_0^t \lambda_1(F_e)(t) dt = \int_0^t \lambda_1(F)(t) dt + \gamma(x, t) - \gamma(x(0), 0)$$

and thus as $t \rightarrow +\infty$ the contraction rate computed from F_e will be the same as the contraction rate computed from F .

These results extend straightforwardly to the weaker forms of contraction discussed above.

Theorem 4. *Consider a non-autonomous system $\dot{x} = f(x, t)$. Assume there exist a metric transformation $\Theta(x, t)$, and a differentiable bounded scalar function $\gamma(t)$, such that*

$$\exists \alpha > 0 \quad \forall t \geq 0 \quad (\lambda_1 + \dots + \lambda_i) + \dot{\gamma} \leq -\alpha$$

where the λ_j are the ranked eigenvalues of the Hermitian part of the generalized Jacobian associated with $\Theta(x, t)$. Then any i -dimensional volume shrinks exponentially to zero with rate α .

The above may also be applied to virtual systems in the sense of [10], which by nature are non-autonomous.

Also note that the system combination properties derived earlier extend straightforwardly to non-autonomous systems.

4 Discussion

Just as contraction can be understood in terms of differential line elements and Riemannian distances, weak contraction can be understood in terms of shrinking of differential planar areas. A key insight of [3] is that, in the case of autonomous systems, weak contraction does not just imply that all surfaces shrink, but actually more precisely that any bounded system trajectory tends to some *equilibrium point*. This is achieved by proving that neither limit cycles or more complex behaviors can occur in that case.

To prove the non-existence of limit cycles [3], assume a that closed curve \mathcal{C} is preserved under the flow of the system, and consider the two-dimensional submanifold of smallest area having \mathcal{C} as a boundary (like a film of soapy water on a loop for blowing bubbles). Since this area must shrink under the flow of the system, this contradicts the preservation of \mathcal{C} . The extension to nonlinear metrics and Riemannian area integrals discussed in this paper is natural.

The non-existence of more complex (chaotic) behaviour follows from the fact that the weak contraction conditions are open – i.e. the strict inequality means they are preserved under small perturbations of f – and the Pugh closing lemma which states, roughly speaking, that any chaotic attractor can be perturbed by a small amount to result in a closed cycle, see [3].

It is interesting to note that transverse contracting systems [11] satisfy $\lambda_1 + \lambda_2 = \lambda_2 < 0$ for some choice of metric. Transverse contracting systems have the property that all solutions converge to the same limit set, which is either a unique equilibrium or a unique limit cycle. This does not contradict the above results, as convergence to a limit cycle can only occur if the domain of transverse contraction is not a simply connected set. In fact, if a system is transverse contracting on a simply connected set, then all solutions converge to a unique equilibrium (note again that transverse contraction is a weaker condition than contraction on the same set, but convergence may only be asymptotic).

Since combination properties are based on algebraic rather than topological considerations, the fact that transverse contracting systems satisfy $\lambda_1 + \lambda_2 < 0$ in some metric also explains the similarity between the combination properties of weakly contracting systems (section 2) and transverse contracting systems ([11], section 4) when connected to a contracting system.

In biology and robotics (e.g. locomotion), natural behaviours of dynamic systems include convergence to an equilibrium (unique or not) and oscillation, so weak contraction and transverse contraction can potentially provide useful frameworks for studying interconnection of biological systems.

Synchronisation is a special case of weak contraction in which $\lambda_1 = 0$ and the rate of convergence to a synchronised state is given by the first non-zero eigenvalue (which, for identical subsystems of dimension m , is λ_{m+1}). Con-

traction behavior of the synchronized system can in turn be analyzed from the reduced (quotient) system, where all synchronized states are collapsed to a single state [10].

The use of norms other than the Euclidean norm in the above results, similarly to the contraction context of [5], is a direction of future research.

References

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